

2 SUSY Lagrangians Part I

2.1 The Free Wess-Zumino Model

Consider a free chiral multiplet

$$S = \int d^4x (\mathcal{L}_s + \mathcal{L}_f) \quad (2.1)$$

$$\mathcal{L}_s = \partial^\mu \phi^* \partial_\mu \phi; \quad \mathcal{L}_f = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi. \quad (2.2)$$

Note: we will be using the metric

$$g^{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad (2.3)$$

Infinitesimal SUSY transformations:

$$\phi \rightarrow \phi + \delta\phi \quad (2.4)$$

$$\psi \rightarrow \psi + \delta\psi \quad (2.5)$$

$$\delta\phi = \epsilon^\alpha \psi_\alpha \quad (2.6)$$

$$= \epsilon^\alpha \epsilon_{\alpha\beta} \psi^\beta \equiv \epsilon\psi \quad (2.7)$$

where

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.8)$$

note:

$$\epsilon\psi = -\psi^\beta \epsilon_{\alpha\beta} \epsilon^\alpha = \psi^\beta \epsilon_{\beta\alpha} \epsilon^\alpha = \psi\epsilon \quad (2.9)$$

$$\delta\phi^* = \epsilon_{\dagger\dot{\alpha}} \psi^{\dagger\dot{\alpha}} \equiv \epsilon^\dagger \psi^\dagger \quad (2.10)$$

$$\delta\mathcal{L}_s = \epsilon \partial^\mu \psi \partial_\mu \phi^* + \epsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi. \quad (2.11)$$

$$\delta\psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \partial_\nu \phi; \quad \delta\psi^\dagger_{\dot{\alpha}} = i(\epsilon \sigma^\nu)_{\dot{\alpha}} \partial_\nu \phi^*. \quad (2.12)$$

$$\delta\mathcal{L}_f = -\epsilon\sigma^\nu\partial_\nu\phi^*\bar{\sigma}^\mu\partial_\mu\psi + \psi^\dagger\bar{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\mu\partial_\nu\phi. \quad (2.13)$$

$$[\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu]_\alpha^\beta = 2\eta^{\mu\nu}\delta_\alpha^\beta; \quad [\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu]_{\dot{\alpha}}^{\dot{\beta}} = 2\eta^{\mu\nu}\delta_{\dot{\alpha}}^{\dot{\beta}} \quad (2.14)$$

$$\begin{aligned} \delta\mathcal{L}_f = & -\epsilon\partial^\mu\psi\partial_\mu\phi^* - \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi \\ & +\partial_\mu\left(\epsilon\sigma^\mu\bar{\sigma}^\nu\psi\partial_\nu\phi^* - \epsilon\psi\partial^\mu\phi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi\right). \end{aligned} \quad (2.15)$$

Thus the action is invariant:

$$\delta S = 0 \quad (2.16)$$

2.2 Commutators of SUSY Transformations

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi = -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu\phi. \quad (2.17)$$

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha = -i(\sigma^\nu\epsilon_1^\dagger)_\alpha\epsilon_2\partial_\nu\psi + i(\sigma^\nu\epsilon_2^\dagger)_\alpha\epsilon_1\partial_\nu\psi. \quad (2.18)$$

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\chi\eta) - (\xi\chi)\eta_\alpha \quad (2.19)$$

$$\begin{aligned} (\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha = & -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu\psi_\alpha \\ & +i(\epsilon_{1\alpha}\epsilon_2^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \epsilon_{2\alpha}\epsilon_1^\dagger\bar{\sigma}^\mu\partial_\mu\psi). \end{aligned} \quad (2.20)$$

Since the last term vanishes on shell, the SUSY algebra closes on shell. What happens off shell?

	Off Shell	On Shell	
ϕ, ϕ^*	2 d.o.f.	2 d.o.f.	(2.21)
$\psi_\alpha, \psi_\alpha^\dagger$	4 d.o.f.	2 d.o.f.	

Book-keeping trick: add an auxillary boson field F

$$\begin{array}{ccc} & \text{Off Shell} & \text{On Shell} \\ F, F^* & 2 \text{ d.o.f.} & 0 \text{ d.o.f.} \end{array} \quad (2.22)$$

$$\mathcal{L}_{\text{aux}} = F^* F. \quad (2.23)$$

$$\delta F = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi; \quad \delta F^* = i\partial_\mu \psi^\dagger \bar{\sigma}^\mu \epsilon. \quad (2.24)$$

$$\delta \mathcal{L}_{\text{aux}} = i\partial_\mu \psi^\dagger \bar{\sigma}^\mu \epsilon F - i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi F^* \quad (2.25)$$

$$\delta \psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \partial_\nu \phi + \epsilon_\alpha F; \quad \delta \psi_\alpha^\dagger = +i(\epsilon \sigma^\nu)_{\dot{\alpha}} \partial_\nu \phi^* + \epsilon_{\dot{\alpha}}^\dagger F^* \quad (2.26)$$

$$S^{\text{new}} = \int d^4x (\mathcal{L}_s + \mathcal{L}_f + \mathcal{L}_{\text{aux}}) \quad (2.27)$$

$$\delta S^{\text{new}} = 0 \quad (2.28)$$

$$X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*, \quad (2.29)$$

$$(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2}) X = -i(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \partial_\mu X \quad (2.30)$$

2.3 The Supercurrent and SUSY Algebra

$$\epsilon J_\mu + \epsilon^\dagger J_\mu^\dagger \equiv \sum_X \delta X \frac{\delta \mathcal{L}}{\delta (\partial^\mu X)} - V_\mu, \quad (2.31)$$

$$\partial^\mu V_\mu = \delta \mathcal{L}. \quad (2.32)$$

$$J_\alpha^\mu = (\sigma^\nu \bar{\sigma}^\mu \psi)_\alpha \partial_\nu \phi^*; \quad J_{\dot{\alpha}}^{\dagger\mu} = (\psi^\dagger \bar{\sigma}^\mu \sigma^\nu)_{\dot{\alpha}} \partial_\nu \phi. \quad (2.33)$$

conserved charges:

$$Q_\alpha = \sqrt{2} \int d^3x J_\alpha^0; \quad Q_{\dot{\alpha}}^\dagger = \sqrt{2} \int d^3x J_{\dot{\alpha}}^{\dagger 0} \quad (2.34)$$

these charges generate the SUSY transformations

$$[\epsilon Q + \epsilon^\dagger Q^\dagger, X] = -i\sqrt{2} \delta X \quad (2.35)$$

for any field X , up to terms which vanish on-shell. This can be verified explicitly by using the canonical equal-time (anti-)commutators for the fields and equations of motion.

Commutators of charges acting on fields give:

$$\begin{aligned} [\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, [\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger, X]] - [\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger, [\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, X]] = \\ 2(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger) i \partial_\mu X \end{aligned} \quad (2.36)$$

$$[[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, \epsilon_1 Q + \epsilon_1^\dagger Q^\dagger], X] = 2(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger) [P_\mu, X], \quad (2.37)$$

$$[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, \epsilon_1 Q + \epsilon_1^\dagger Q^\dagger] = 2(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger) P_\mu \quad (2.38)$$

Extracting ϵ and ϵ^\dagger gives:

$$\{Q_\alpha, Q_{\dot{\alpha}}^\dagger\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu, \quad (2.39)$$

$$\{Q_\alpha, Q_\beta\} = \{Q_{\dot{\alpha}}^\dagger, Q_{\dot{\beta}}^\dagger\} = 0 \quad (2.40)$$

2.4 The Interacting Wess-Zumino Model

Consider a collection of free chiral supermultiplets containing ϕ_a , ψ_a , and F_a

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{*a} \partial_\mu \phi_a + i\psi^{\dagger a} \bar{\sigma}^\mu \partial_\mu \psi_a + F^{*a} F_a \quad (2.41)$$

$$\delta\phi_a = \epsilon\psi_a \quad \delta\phi^{*a} = \epsilon^\dagger\psi^{\dagger a} \quad (2.42)$$

$$\delta(\psi_a)_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi_a + \epsilon_\alpha F_a \quad \delta(\psi^{\dagger a})_{\dot{\alpha}} = i(\epsilon\sigma^\mu)_{\dot{\alpha}}\partial_\mu\phi^{*a} + \epsilon_{\dot{\alpha}}^\dagger F^{*a} \quad (2.43)$$

$$\delta F_a = -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi_a \quad \delta F^{*a} = i\partial_\mu\psi^{\dagger a}\bar{\sigma}^\mu\epsilon. \quad (2.44)$$

The most general set of renormalizable interactions for these fields is

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}W^{ab}\psi_a\psi_b + W^a F_a + h.c., \quad (2.45)$$

where for renormalizability W^{ab} is a linear function of ϕ and ϕ^* ; W^a is a quadratic function of ϕ and ϕ^* . Note: W^{ab} is symmetric under $a \leftrightarrow b$.

Adding a function of ϕ_a, ϕ^{*a} would break SUSY.

$$\delta\mathcal{L}_{\text{int}}|_{4\text{-spinor}} = -\frac{1}{2}\frac{\delta W^{ab}}{\delta\phi_c}(\epsilon\psi_c)(\psi_a\psi_b) - \frac{1}{2}\frac{\delta W^{ab}}{\delta\phi^{*c}}(\epsilon^\dagger\psi^{\dagger c})(\psi_a\psi_b) + h.c. \quad (2.46)$$

The Fierz identity eq. (2.19) implies

$$(\epsilon\psi_a)(\psi_b\psi_c) + (\epsilon\psi_b)(\psi_c\psi_a) + (\epsilon\psi_c)(\psi_a\psi_b) = 0, \quad (2.47)$$

so $\delta\mathcal{L}_{\text{int}}$ vanishes if and only if $\delta W^{ab}/\delta\phi_c$ is totally symmetric under interchange of a, b, c . We also need $\delta W^{ab}/\delta\phi^{*c} = 0$, so W^{ij} is *analytic* (or *holomorphic*) in the complex fields ϕ_c .

It is convenient to write

$$W^{ab} = \frac{\delta^2}{\delta\phi_a\delta\phi_b}W \quad (2.48)$$

where

$$W = E^a\phi_a + \frac{1}{2}M^{ab}\phi_a\phi_b + \frac{1}{6}y^{abc}\phi_a\phi_b\phi_c \quad (2.49)$$

and M^{ab} , y^{abc} are mass and Yukawa matrices which are symmetric under interchange of indices. W is called the *superpotential*.

$$\delta\mathcal{L}_{\text{int}}|_{\partial} = -iW^{ab}\partial_\mu\phi_b\psi_a\sigma^\mu\epsilon^\dagger - iW^a\partial_\mu\psi_a\sigma^\mu\epsilon^\dagger + h.c. \quad (2.50)$$

Note:

$$W^{ab}\partial_\mu\phi_b = \partial_\mu\left(\frac{\delta W}{\delta\phi_a}\right). \quad (2.51)$$

so Eq. (2.50) will be a total derivative if and only if

$$W^a = \frac{\delta W}{\delta \phi_a} \quad (2.52)$$

The remaining terms then cancel.

We have found that all renormalizable non-gauge interactions are determined by a single function W which is holomorphic in ϕ^a . Our demonstration that the Action was a SUSY invariant did not rely on the precise form of the superpotential, only that it was holomorphic. A general holomorphic superpotential will give a supersymmetric theory with non-renormalizable interactions (it will not however have the most general set of non-renormalizable interactions allowed by supersymmetry).

The action is quadratic in F

$$\mathcal{L}_F = F_a F^{*a} + W^a F_a + W_a^* F^{*a} \quad (2.53)$$

so we can integrate it out exactly

$$F_a = -W_a^*; \quad F^{*a} = -W^a. \quad (2.54)$$

$$\begin{aligned} \mathcal{L} = & \partial^\mu \phi^{*a} \partial_\mu \phi_a + i \psi^{\dagger a} \bar{\sigma}^\mu \partial_\mu \psi_a \\ & - \frac{1}{2} \left(W^{ab} \psi_a \psi_b + W^{*ab} \psi^{\dagger a} \psi^{\dagger b} \right) - W^a W_a^*. \end{aligned} \quad (2.55)$$

For the time being we will take $E^a = 0$, we will consider non-zero values when we discuss spontaneous SUSY breaking. The scalar potential is then given by:

$$\begin{aligned} V(\phi, \phi^*) = & W^a W_a^* = F_a F^{*a} = M_{ac}^* M^{cb} \phi^{*a} \phi_b \\ & + \frac{1}{2} M^{ad} y_{bcd}^* \phi_a \phi^{*b} \phi^{*c} + \frac{1}{2} M_{ad}^* y^{bcd} \phi^{*a} \phi_b \phi_c + \frac{1}{4} y^{abd} y_{ced}^* \phi_a \phi_b \phi^{*c} \phi^{*e}. \end{aligned} \quad (2.56)$$

$$V \geq 0 \quad (2.57)$$

$$\begin{aligned} \mathcal{L}_{\text{WZ}} = & \partial^\mu \phi^{*a} \partial_\mu \phi_a + i \psi^{\dagger a} \bar{\sigma}^\mu \partial_\mu \psi_a \\ & - \frac{1}{2} M^{ab} \psi_a \psi_b - \frac{1}{2} M_{ab}^* \psi^{\dagger a} \psi^{\dagger b} - V(\phi, \phi^*) \\ & - \frac{1}{2} y^{abc} \phi_a \psi_b \psi_c - \frac{1}{2} y_{abc}^* \phi^{*a} \psi^{\dagger b} \psi^{\dagger c}. \end{aligned} \quad (2.58)$$

linearized equations of motion:

$$\partial^\mu \partial_\mu \phi_a = -M_{ac}^* M^{cb} \phi_b + \dots; \quad (2.59)$$

$$i\bar{\sigma}^\mu \partial_\mu \psi_a = M_{ab}^* \psi^{\dagger b} + \dots; \quad i\sigma^\mu \partial_\mu \psi^{\dagger a} = M^{ab} \psi_b + \dots \quad (2.60)$$

using Eq. (2.14) we can obtain

$$\partial^\mu \partial_\mu \psi_a = -M_{ac}^* M^{cb} \psi_b + \dots; \quad \partial^\mu \partial_\mu \psi^{\dagger b} = -\psi^{\dagger a} M_{ac}^* M^{cb} + \dots \quad (2.61)$$

Therefore the scalars and fermions have the same mass eigenvalues, diagonalizing gives a collection of massive chiral supermultiplets.

References

- [1] S. Martin, “A Supersymmetry Primer”, hep-ph/9709356.
- [2] J. Wess and B. Zumino, *Nucl. Phys.* **B70**, 39 (1974).